1.6 Summary of Chapter 1

We prove that there are infinitely primes in two ways, by showing the divergences of either

$$\sum_{p\le N}\frac{1}{p}\ \text{as}\ N\to\infty \qquad \text{or} \qquad \sum_{p}\frac{1}{p^\sigma}\ \text{as}\ \sigma\to 1+\,.$$

In both cases primes are identified by the fact that integers can be factorised into primes, unique apart from order.

The important technique in this chapter is estimating sums by integrals.

The important objects introduced in this chapter are the Riemann zeta function, Euler products and their connection.

It might be of interest to compare Theorem 1.4

$$\sum_{p\le N}\frac{1}{p} > \log\log\left(N+1\right)-1 \quad \text{with} \quad \sum_{n\le N}\frac{1}{n} > \log\left(N+1\right)$$

of Theorem 1.2, or Theorem 1.13

$$\sum_{p}\frac{1}{p^\sigma} > \log\left(\frac{1}{\sigma-1}\right)-1 \quad \text{with} \quad \sum_{n=1}^{\infty}\frac{1}{n^\sigma} \ge \frac{1}{\sigma-1}$$

of Theorem 1.7. The estimates of the sum over primes is approximately the logarithm of the size for the sums over integers. Perhaps this is connected to the Prime Number Theorem, proved later in this course, which states that

$$\sum_{p\le x} 1 \sim \frac{x}{\log x},$$

as $x\to\infty$, which can be compared with $\sum_{n\le x} 1\sim x$.

Here $f(x)\sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x)=1$.